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THE TWO-FLUID MODEL OF FLOW OF GAS WITH PARTICLES DISPERSED IN IT*

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Several problems of the theory of gas flow with dispersed solid particles of small but finite volume are considered using the two-fluid model. Integral laws of conservation for a mixture and particles, supplemented by formulas for the force of interaction and for the heat flux between the media, and by equations of state, are used as the basis of the analysis. The gas viscosity and thermal conductivity are assumed, as in /1/, of any significance only in relation to gas interaction with particles. Interaction between particles is taken into account only at discontinuity surfaces of the "film" /2—4/ and "cluster" /5/ types. In analyzing discontinuity surfaces within the scope of the model additional assumptions are made about their structure, which are required for closing the system of conditions at the discontinuity. The self-similar problem about the initial disintegration stage of an arbitrary discontinuity is investigated.

1. Omitting the listing of assumptions that are made in two-fluid models of interpenetrating continuous media /1- 4, 6- 9/, we present below the integral laws of conservation for the mixture and particles.

We use the notation: p for the pressure of gas, ρ°, e, $i = e + p/\rho$, T and $\mathbf{v}$ for the gas true density, internal specific energy and enthalpy, temperature and velocity, respectively, with a ρ_s°, e_s, T_s and $\mathbf{v}_s$ for the respective parameters of "gas" of particles (the second phase). We also introduce $i_s = e_s + \varepsilon p$ with $\varepsilon = 1/\rho_s^\circ$, although owing to the assumed absence of proper pressure of gas of particles, i_s is not the specific enthalpy of a particles. Besides ρ° and ρ_s° we use "blurred" densities ρ and ρ_s, respectively, the masses of gas and particles per unit of mixture volume, and their porosities $m = \rho/\rho^\circ$ and $m_s = \rho_s/\rho_s^\circ$ with

$$m + m_s \equiv (\rho/\rho^\circ) + (\rho_s/\rho_s^\circ) = 1 \tag{1.1}$$

Denoting the arbitrary time-independent volume occupied by the mixture by Ω, its bounding surface by $\partial\Omega$, and by $d\sigma$ an element of $\partial\Omega$, with $\mathbf{n}$ representing the external unit vector, we have for the investigated flow, in the absence of external forces and energy sources the following integral conservation laws:

$$\iiint_\Omega \rho\, d\Omega \Big|_{t_0}^{t} = -\int_{t_0}^{t} dt \iint_{\partial\Omega} \rho v_n\, d\sigma \quad (v_n = \mathbf{v}\cdot\mathbf{n}) \tag{1.2}$$

$$\iiint_\Omega (\rho\mathbf{v} + \rho_s\mathbf{v}_s)\, d\Omega \Big|_{t_0}^{t} = -\int_{t_0}^{t} dt \iint_{\partial\Omega} (p\mathbf{n} + \rho v_n\mathbf{v} + \rho_s v_{sn}\mathbf{v}_s)\, d\sigma$$

$$\iiint_\Omega \{\rho(2e + v^2) + \rho_s(2e_s + v_s^2)\}\, d\Omega \Big|_{t_0}^{t} = -\int_{t_0}^{t} dt \iint_{\partial\Omega} \{\rho v_n(2i + v^2) + \rho_s v_{sn}(2i_s + v_s^2)\}\, d\sigma \quad (v = |\mathbf{v}|)$$

$$\iiint_\Omega \rho_s\, d\Omega \Big|_{t_0}^{t} = -\int_{t_0}^{t} dt \iint_{\partial\Omega} \rho_s v_{sn}\, d\sigma$$

$$\iiint_\Omega \rho_s\mathbf{v}_s\, d\Omega \Big|_{t_0}^{t} = -\int_{t_0}^{t} dt \left\{\iint_{\partial\Omega} (m_s p\mathbf{n} + \rho_s v_{sn}\mathbf{v}_s)\, d\sigma - \iiint_\Omega \boldsymbol{\varphi}\, d\Omega\right\}$$

$$\iiint_\Omega \rho_s(2e_s + v_s^2)\, d\Omega \Big|_{t_0}^{t} = -\int_{t_0}^{t} dt \left\{\iint_{\partial\Omega} \rho_s v_{sn}(2i_s + v_s^2)\, d\sigma - 2\iiint_\Omega (\rho_s q + \boldsymbol{\varphi}\mathbf{v}_s)\, d\Omega\right\}$$

where $t > t_0$ with t_0 and t denoting aribtrary instants of time, $\boldsymbol{\varphi}$ is the force exerted by gas

*Prikl.Matem.Mekhan.,46,No.1,96-106,1982 on particles in a unit volume, and $\rho_s q$ is the respective heat flux to particles. To close (1.1) and (1.2) it is necessary to know the equations of state and the expressions for φ and q which we assume to be

$$e = e(p, T), \quad \rho^\circ = \rho^\circ(p, T), \quad e_s = e_s(T_s), \quad \varphi = p\nabla m_s + \rho_s \mathbf{f}, \quad \mathbf{f} = \varphi_f \cdot (\mathbf{v} - \mathbf{v}_s), \quad q = \varphi_q \cdot (T - T_s) \tag{1.3}$$

with φ_f and φ_q taken to be known functions of thermodynamic parameters and of the absolute value of the relative particle velocity $\mathbf{v} - \mathbf{v}_s$, but not of their derivatives. The latter implies the disregard of the effect of additional mass. The term $p\nabla m_s = -p\nabla m$ in φ takes into account the variability of the volume occupied by particles or, what is the same, variation of the "through flow" cross section of the space free of particles.

Integral laws of the form (1.2) are not the only ones possible. Thus, the equations of motion and energy of the mixture can be replaced by similar equations for the gas. The advantage of the representation used here is that (1.2) contains the maximum number of equations in which φ and q are absent, hence not associated with assumptions about the interphase interaction mechanism. This is particularly convenient in the analysis of discontinuities. In the absence of particle interaction, which in (1.2) is assumed everywhere, except possibly on some surfaces, there is no exchange of particle kinetic and internal energies. If such exchange is absent throughout Ω, the last of Eqs.(1.2) can be replaced by the law of conservation of particle internal energy

$$\iiint_\Omega \rho e_s \, d\Omega \,\Big|_{t_0}^{t} = -\int_{t_0}^{t} dt \left(\iint_{\partial\Omega} \rho_s v_{sn} e_s \, d\sigma - \iiint_\Omega \rho_s q \, d\Omega \right) \tag{1.4}$$

In subregions of parameter continuity we obtain from (1.2) with allowance for (1.1) and (1.3) the following differential equations which it is convenient for the subsequent analysis to represent together with (1.1) as

$$\begin{gathered}
\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0, \quad \frac{D\mathbf{v}}{Dt} + \frac{1}{\rho^\circ}\nabla p + \frac{\rho_s}{\rho}\mathbf{f} = 0 \\
T\frac{Ds}{Dt} = \frac{Di}{Dt} - \frac{1}{\rho^\circ}\frac{Dp}{Dt} = -N, \quad \frac{\partial \rho_s}{\partial t} + \nabla(\rho_s \mathbf{v}_s) = 0 \\
\frac{D^s \mathbf{v}_s}{Dt} + \varepsilon \nabla p = \mathbf{f}, \quad \frac{D^s e_s}{Dt} = q, \quad \rho = \rho^\circ - \rho_s \delta \\
\left(\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}\nabla, \quad \frac{D^s}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_s \nabla, \right. \\
\left. N = \frac{\rho_s}{\rho}\{q + (\mathbf{v}_s - \mathbf{v})\mathbf{f}\}, \quad \varepsilon = \frac{1}{\rho_s^\circ}, \quad \delta = \frac{\rho^\circ}{\rho_s^\circ} \right)
\end{gathered} \tag{1.5}$$

where D/Dt and D^s/Dt are operators of total differentiation with respect to t along the gas and particle trajectories, and s is the specific entropy of gas.

The fifth and sixth of Eqs.(1.5) are usually postulated, not derived from integral laws, and supplemented by expressions (1.3) for φ and q. Besides the usual considerations on the role of integral and differential forms of the laws of conservation, the method adopted here is preferable for the following reason. The term $p\nabla m_s$ in (1.3), which, incidentally, is taken into account in the theory of gas filtration in a medium of variable porosity, is more reasonable that $\varepsilon\nabla p$ in the equation of motion of a single particle, and the fifth of Eqs.(1.5) is just such equation that can be taken as the input one. Terms containing ε and δ in (1.5) correspond to terms $m_s = \rho_s / \rho_s^\circ$ in (1.1) and $p\nabla m_s$ in (1.3) which in our model appear inter-changeably as terms that take into account the volume of particles.

2. The strong discontinuities admitted by (1.2) belong to one of the following two classes: discontinuities of the film (cluster) type that carry surface mass, momenta, etc., and those do not have such properties. We begin the analysis on discontinuities of the second class, and develop and refine the results of /1,3/.

At the discontinuity $\partial\Omega_d$, generally m_s becomes discontinuous besides other parameters, hence $(\nabla m_s)_n$ and the normal to the discontinuity component of φ become infinite. This inhibits the use of the expression for φ_n from (1.3) and shows the necessity of introducing the normal to $\partial\Omega_d$ component F_n of the surface force $\mathbf{F}$ (surface quantities are denoted below by capital letters). Since the formula for φ is inapplicable on $\partial\Omega_d$, the surface force tangent component $\mathbf{F}_\tau$ may generally appear in that case. The model of the medium must include, besides (1.3), either the definition of F_n and $\mathbf{F}_\tau$ or the formulation of equivalent, physically valid assumptions generally specific for various discontinuities.

Let D be the discontinuity velocity along the normal $\mathbf{n}$ to $\partial\Omega_d$ and $[\varphi] = \varphi_+ - \varphi_-$ the difference between φ on various sides of it. At "mass-free" discontinuities it is reasonable to neglect particle interaction, which would enable us to use the law of conservation of particle energy in the form (1.4). From (1.2) and (1.4) then follows that at the investigated discontinuities the following conditions are satisfied:

$$\Delta^j \equiv [\rho(v_n - D)] = 0, \quad \Delta^n \equiv [p + j(v_n - D) + j_s(v_{sn} - D)] = 0 \tag{2.1}$$
$$\Delta^\tau \equiv [j\mathbf{v}_\tau + j_s\mathbf{v}_{s\tau}] = 0, \quad \Delta^i \equiv [j\{2i + (v_n - D)^2 + v_\tau^2\} +$$
$$j_s\{2i_s + (v_{sn} - D)^2 + v_{s\tau}^2\}] = 0, \quad \Delta_s{}^j \equiv [\rho_s(v_{sn} - D)] = 0$$
$$\Delta_s{}^n \equiv [m_s p] + j_s[v_{sn}] - F_n = 0, \quad \Delta_s{}^\tau \equiv j_s[\mathbf{v}_{s\tau}] - \mathbf{F}_\tau = 0$$
$$\Delta_s{}^e \equiv j_s[e_s] = 0 \quad (j = \rho_\pm(v_{n\pm} - D))$$

where j and j_s are stream densities continuous on $\partial\Omega_d$, and $v_\tau = |\mathbf{v}_\tau|$. The distinction between (2.1) and conditions in /1/ is due to the form of the sixth and seventh conditions with the, so far, unknown F_n and $\mathbf{F}_\tau$.

It was assumed in /1/ that $\mathbf{F}_\tau = 0$ and the equality

$$[(v_{sn} - D)^2] + 2\varepsilon[p] = 0 \tag{2.2}$$

was substituted for the sixth condition in (2.1). Condition (2.2) in /1/ was obtained by integrating the fifth of Eqs. (1.5) with $\mathbf{f} = 0$ over the "blurred" discontinuity stationary in the corresponding coordinate system (condition (2.2) was considered in /10/ prior to /1/). Hence if (2.2) is to be valid it is necessary (but not sufficient, as shown below) that particles intersect $\partial\Omega_d$. This was not taken into account in the investigation of discontinuities in /1/, which resulted in a number of inaccuracies that were repeated in /7,9,11/.

According to (2.1) five types of discontinuities are possible. When $j = j_s = 0$, the expressions for F_n and $\mathbf{F}_\tau$ are not required. Here, by virtue of (2.1), we have

$$[p] = 0, \quad v_{n\pm} - D = v_{sn\pm} - D = 0, \quad \mathbf{F}_\tau = 0, \quad F_n = p\,[m_s] \tag{2.3}$$

The discontinuities of $\mathbf{v}_\tau$, $\mathbf{v}_{s\tau}$, m_s, e_s and T are arbitrary and with allowance for (1.1), (1.3), and (2.3) define discontinuities of other parameters. In this case we have a contact (tangential) discontinuity in both media.

When $j = 0$ and $j_s \neq 0$, the closure of (2.1) requires, unlike in the previous case, the determination of $\mathbf{F}_\tau$. Let us set $\mathbf{F}_\tau = 0$ which is now an assumption, not a corollary of (2.1). When $\mathbf{F}_\tau = 0$, conditions $\Delta^i = 0, \Delta_s{}^\tau = 0$ and $\Delta_s{}^e = 0$ lead to equality (2.2) from which and other conditions (2.1) follows the continuity of p and all other parameters of particles and also $v_{n\pm} - D = 0$. Discontinuity of $\mathbf{v}_\tau$ and T are arbitrary, and the discontinuities of other parameters of gas are obtained in terms of $[T]$, $[p] = 0$ and $[m_s] = 0$ from (1.1) and (1.3). Here we have in the gas a contact discontinuity intersected by a continuous stream of particles, along which, by virtue of (2.1) $F_n = 0$

If the gas flows through a contact discontinuity in the medium of particles $(j \neq 0, j_s = 0)$, then, by virtue of (2.1) $\mathbf{F}_\tau = 0$, and it is necessary to have either the expression for F_n or a supplementary assumption on the relations between the parameters of gas. In the second case F_n, whenever necessary, is determined in conformity with (2.1) using the formula $F_n = [m_s p]$. To formulate such assumptions or expressions we used data from /12/ relative to flows in channels with abrupt area changes. Let $m_{s-} < m_{s+}$, $|v_{n\mp} - D| \leqslant a_\mp$ and a be the speed of sound in the gas. Then, if the gas flows from the region whose parameters are denoted by the subscript minus, we set

$$[s] = 0 \tag{2.4}$$

When $m_{s-} > m_{s+}$ and $|v_{n-} - D| < a_-$, the application of the "Borda" scheme yields

$$F_n = [m_s]p_- \tag{2.5}$$

Prior to the appearance of /12/ relations (2.4) and (2.5) were used for analyzing flows in channels with abrupt changes of area, and shock wave interaction with porous barriers /13/.

If $m_{s-} > m_{s+}$ and $|v_{n-} - D| \geqslant a_-$, an additional conditions for closing system (2.1) is required, as in /12/, only when $|v_{n+} - D| > a_+$, and is of the form

$$F_n = [m_s]p_+ \tag{2.6}$$

When $-[m_s] \ll 1$, the substitution of (2.4) for (2.6) does not greatly affect the results. The case of $m_{s-} < m_{s+}$ and $|v_{n\mp} - D| > a_\mp$ — not considered here— requires special analysis, although it is possible to apply (2.4) when $[m_s] \ll 1$.

Using one of conditions (2.4)—(2.6) together with (1.1), (1.3), the equality $F_n = [m_s p]$, and when $j_s = 0$, the implied by (2.1) relations

$$[\mathbf{v}_\tau] = 0, \quad \rho_+ / \rho_- = (v_{n-} - D) / (v_{n+} - D), \quad [v_n - D] = -[p] / j$$

$$2 [i] = (\rho_+^{-1} + \rho_-^{-1})[p], \quad [p] = -j^2 [\rho^{-1}]$$

provides the complete system of conditions that links gas parameters at the contact discontinuity of particles. Here $(v_{sn\pm} - D) = 0$, with arbitrary $[m_s]$, $[\mathbf{v}_{s\tau}]$ and $[e_s]$.

The last case of $j \neq 0$ and $j_s \neq 0$ is the most complicated, since it requires supplementary information about F_n and $\mathbf{F}_\tau$. Without attempting at its exhaustive investigation, we shall consider only two fairly typical situation. We begin by stating that for small m_s for the determination of F_n and $\mathbf{F}_\tau$ or of variations of component $\mathbf{v}_s$ it is reasonable to consider the problem of intersection of the stationary discontinuity plane by an isolated particle of radius r. Let l be the relaxation length of the particle velocity lag. It can be shown that for the Stokes law of drag $l = 2r^2\rho_s^\circ v_{sn} / (9\nu\rho^\circ)$, where ν is the coefficient of the gas kinematic viscosity. Let us consider the case in which r and the discontinuity width λ are considerably small than l. As a rule, condition $r \ll l$ is satisifed, since the Reynolds number $\mathrm{Re}_s \equiv r v_{sn} / \nu \gtrsim 1$, while $\rho_s^\circ / \rho^\circ \gg 1$. The inequality $\lambda \ll l$ is, mostly also valid. Indeed, even in the case of weak compression shocks of comparatively large thickness $\lambda \sim \nu / \{a (M_n - 1)\}$, where $M_n = v_n / a$, and the ratio $l / \lambda \sim (M_n - 1)\mathrm{Re}_s^2\rho_s^\circ / \rho^\circ$, hence for $\mathrm{Re}_s \gtrsim 1$ and $M_n - 1 \gg \rho^\circ / \rho_s^\circ$, which is usually the case, we have $l / \lambda \gg 1$.

The introduction of constraints in the analysis of particle movement through the discontinuity it is not necessary to take into account force $\mathbf{f}$. It seems reasonable to begin the investigation on the case of $r \ll \lambda$ in which for small r and $M_n \to 1$ is always realized, at least in the case of compression shocks. When $r \ll \lambda$ a particle which intersects the discontinuity moves in a continuous stream whose variation depends only on $\mathbf{n}$. Taking this into account, we integrate the fifth of Eqs.(1.5) for $\mathbf{f} = 0$, defined in a system in which at the considered instant of time the discontinuity is quiescent (the discontinuity is assumed to have a stationary structure and λ / v_{sn} to be considerably less than the characteristic variation of its velocity D) , we obtain (2.2) and

$$[\mathbf{v}_{s\tau}] = 0 \tag{2.7}$$

which together with (2.1) imply that $\mathbf{F}_\tau = 0$, and the remaining equations (2.1) are equivalent to the system

$$[\rho (v_n - D)] = 0, \quad [p + j (v_n - D) + j_s (v_{sn} - D)] = 0 \ , \ [\mathbf{v}_\tau] = 0 \tag{2.8}$$

$$[2i + (v_n - D)^2] = 0, \quad [\rho_s (v_{sn} - D)] = 0, \quad [e_s] = 0 \ , \ F_n = [m_s p + j_s (v_{sn} - D)]$$

The last of Eqs.(2.8) determines F_n after variations of all parameters have been determined by the remaining conditions (2.8) supplemented by (2.2), (2.7), (1.1), and (1.3). Here, as in the case of shock waves, when parameters D, j, j_s ahead of the discontinuity are known, the parameters behind it are determined by specifying p_+ or some other quantity that defines the discontinuity intensity. As $[p] \to 0$, when $j \neq 0$ and $j_s \neq 0$, the above applies to discontinuities of all parameters. The limit velocity D (as $[p] \to 0$) is determined by the condition of existence of a nontrivial solution of an algebraic system that is linear and homogeneous with respect to $[p], \ldots$, and is the result of linearization of (2.2) and (2.8) with allowance for (1.1) and (1.3). As $[p] \to 0$, velocity D satisfies the equation

$$f (X, \Delta, \chi) \equiv (X - \Delta)^2(X^2 - 1) - \chi X^2 = 0 \tag{2.9}$$

$$X = (v_n - D) / A, \quad \Delta = (v_n - v_{sn}) / A, \quad \chi = \rho_s\varepsilon\delta / (1 - \varepsilon\rho_s) = \rho_s\rho^{\circ 2} / (\rho\rho_s^{\circ 2}), \quad A^2 = a^2\rho^{\circ 2} (1 - \varepsilon\rho_s) / \{\rho\rho^\circ + \alpha\rho_s (\delta - \varepsilon\rho^\circ)a^2\}$$

where $\alpha T = (\partial\rho^\circ / \partial s)_p$. If $\varepsilon = \delta / \rho^\circ = 1 / \rho_s^\circ$, which is the case with our model, then $A = a$. As shown in /1/, (2.9) with $A = a$ defines also the characteristics of one-dimensional unsteady flows, different from the trajectories of gas and particles.

The number of real roots of Eqs.(2.9) with $\chi > 0$ depends on parameter $|\Delta|$. This is illustrated in Fig.1, where the solid and dash lines represent schematically the first term of (2.9) and χX^2 , respectively, as functions of X when $\Delta \geqslant 0$. Diagrams a-f in Fig.1 correspond to $\Delta = 0$, $0 < \Delta < 1$, $\Delta = 1$, $1 < \Delta < \Delta_* = (1 + \chi^{1/3})^{3/2}$, $\Delta = \Delta_*$ and $\Delta > \Delta_*$ Intersections of solid and dash lines represent roots of Eq.(2.9). It will be seen that there are always at least two real roots. When $\chi \ll 1$, which happens in most cases, they are close to ∓ 1.

In conformity with (2.9)

$$(v_n - D)_{1,2} = X_{1,2}a \approx \mp a\sqrt{1+\chi/(1\pm\Delta)^2}$$
$$(v_{sn} - D)_{1,2} \approx \mp a\{\sqrt{1+\chi/(1\pm\Delta)^2} \pm \Delta\} \qquad (2.10)$$

Formulas with the lower sign at $\Delta > 0$ hold when $(1-\Delta)^2 \gg \chi$. Here and subsequently $\Delta > 0$ corresponds to lagging particles, while $\Delta < 0$ to opposite situation. In the case of $v_n < 0$ the above definition differs from the generally accepted one.

Besides the derived above equations, (2.9) can have other real roots. When $\Delta = 0$, the solid and dash lines are tangent at the coordinate origin, and we have multiple root $X = 0$. Since it corresponds to the already investigated case of $j = j_s = 0$, it is of no interest. If $0 < \Delta < \Delta_*$, (2.9) has no real roots different from $X_{1,2}$. When $\Delta = \Delta_*$, the maximum of the solid curve is tangent to the dash-line parabola, and when $\Delta > \Delta_*$, it intersects the latter. Hence, when $\Delta > \Delta_*$, there are four real roots. Although just before the tangency instant $X_2 > \Delta$, it is convenient to denote by X_2 with $\Delta = \Delta_*$ the root nearest to $X = 1$, i.e. smaller than Δ. Incidentally, formulas (2.10) are valid for quantities with subscript "2" when $(1-\Delta)^2 \gg \chi$ only in this case. By numbering the two remaining roots in the order of their growth we obtain $X_2 < X_3 < \Delta$ and $X_4 > \Delta$. If $\chi \ll 1$, then X_3 and X_4 are close to Δ and to each other. When $\chi/(\Delta^2 - 1) \ll 1$, the following formulas hold:

$$(v_n - D)_{3,4} = X_{3,4}a \approx a\Delta\{1 \mp \sqrt{\chi/(\Delta^2-1)}\} \qquad (2.11)$$
$$(v_{sn} - D)_{3,4} \approx \mp a\sqrt{\chi/(\Delta^2-1)}$$

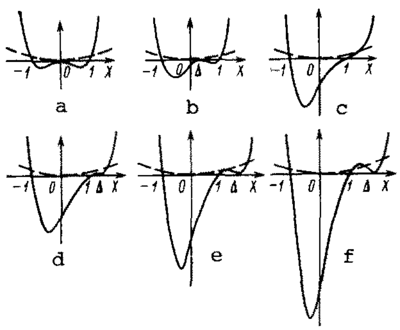

Fig.1

We call the discontinuities that correspond at their low intensity to roots $X_{1,2}$, $c^\pm$-shocks. Weak $c^\pm$-shocks move relative to gas either to the right or left at velocities close to the speed of sound, to consider them as analogs of compression shocks. A further argument in support of this is provided by the calculation of gas entropy increase of such shocks. We call the discontinuities that correspond to roots $X_{3,4}$, $c_s^\pm$-shocks. Weak $c_s^\pm$-shocks slightly outdistance particles (c_s^+) or lag behind them (c_s^-) and together with particles lag behind the gas. The shock-like variation of the c^--shock at transition through $\Delta = \Delta_*$ should not be taken as due only to the root numbering convention. Only with such numbering X_2, while remaining the root closest to $X = 1$ for all Δ, corresponds to a weak c^--shock propagating to the left relative to gas at almost the speed of sound.

In some of the discontinuities considered here the signs of j and j_s differ, i.e. the gas and particles intersect discontinuities from different sides. When $\Delta > 0$ and $\chi \ll 1$, then in conformity with (2.10) and (2.11) this occurs in the case of weak c^--shocks, roughly speaking, when particles lag at supersonic velocity, and in that of c^+_s-shocks always. Since $f(-X, -\Delta, \chi) = f(X, \Delta, \chi)$, this analysis is readily applicable to the case when particles do not lag behind, but outdistance the gas.

We obtain further information about discontinuities by determining the entropy increment in them. Disregarding $\varepsilon = \delta/\rho^\circ$ and using (2.2), (2.8), (1.1), and (1.3), for small $[p]$ we obtain

$$[s] = \left(\frac{\delta_+\rho_+\rho_{s-}}{\rho_+^\circ} + \frac{\delta_-\rho_-\rho_{s+}}{\rho_-^\circ} - 2\,\frac{\rho_+ + \rho_-}{\rho_{s+}+\rho_{s-}}\,\varepsilon\rho_{s+}\rho_{s-}\right)\frac{[p]}{\rho_+\rho_-(T_+ + T_-)}$$

Then, setting $\varepsilon = \delta/\rho^\circ = 1/\rho_s^\circ$, we find that $[s] \sim [p]^3$. Retaining in expansions terms to $[p]^3$, we finally obtain the formula ($V = 1/\rho^\circ$ is the specific volume)

$$[s] = \left(\frac{\rho^\circ\rho_s{}^2a^2 - \rho_s{}^\circ m j_s{}^2}{\rho^2\rho_s{}^{\circ 3}a^2 j_s{}^4}\rho_s{}^3 + \frac{V_{pp}}{3}\right)\frac{[p]^3}{4T}, \quad V_{pp} = \left(\frac{\partial^2 V}{\partial p^2}\right)_s \qquad (2.12)$$

In the absence of particles ($\rho_s = 0$) this formula is the same as that for weak shocks in gas. Substituting $j_s = \rho_s(v_{sn} - D)_{1,2}$ into (2.12) and using (2.9) and (2.10), we find that when $|\Delta| \equiv |v - v_s|/a$ is not close to unity, the first term in parentheses in (2.12) is equal

$$\rho_s\{\rho^\circ - \rho_s{}^\circ m(1\pm\Delta)^2\}/\{a^4\rho^2\rho_s{}^{\circ 3}(1\pm\Delta)^4\}$$

Owing to $\rho_s^{\circ 3}$ in the denominator its absolute value is less than $V_{pp}/3$. Consequently, when $V_{pp} > 0$, weak $c^\pm$-shocks can only be, as in gas, compression shocks in which p and s increase in the direction of the flow of gas.

After the determination of j_s using (2.11) in the case of weak $c_s^{\pm}$-shocks we obtain for the first term in parentheses in (2.12) the expression

$$\{\rho_s^\circ(\Delta^2-1)^2-\Delta^2(\Delta^2-1)\rho_s\}/(\rho_s\rho^{\circ 3}a^4\Delta^4)$$

which in typical situations $(\rho_s^\circ \gg \rho_s)$ is positive and dominates $V_{pp}/3$. Since for any V_{pp}, p and s in weak $c_s^{\pm}$-shocks increase in the direction of gas flow through them. The latter is independent of whether particles lag behind $(\Delta > 1)$ or outdistance $(\Delta < -1)$ the gas.

As previously noted, $c_s^{\pm}$ -shocks are only possible when $\chi \neq 0$ while in the case of $\chi \to 0$ they continuously degenerate into contact discontinuities of the particle medium. Under typical conditions, when $\chi \approx \rho_s\varepsilon\delta \ll 1$, $c_s^{\pm}$ -shocks cannot be intensive. Although this is not a justification for the formula used above for λ, valid in the case of weak shock waves, the inequality $r \ll \lambda$ and, consequently, also equalities (2.2) and (2.7) seem to be at least plausible for $c_s^{\pm}$-shocks. By contrast, as the intensity of c^{+}-shocks increases r and λ are, first, of the same order and, then, the shock thickness becomes much smaller than r. Under such conditions the variation of $\mathbf{v}_s$ can be determined by solving the problem of passage of a sphere through a compression shock. To solve it, it is convenient to use a system of coordinates in which the unperturbed shock at the considered instant (in the previously defined sense) is at rest, and the velocity of gas has only its normal component. In that system the particle velocity component tangent to the shock prior to that instant is equal $\mathbf{v}_{s\tau}-\mathbf{v}_\tau$. We denote by subscript 1 the projections of $\mathbf{v}_\tau$, $\mathbf{v}_{s\tau}$ and $\mathbf{F}_\tau$ on the direction of $\mathbf{v}_{s\tau}-\mathbf{v}_\tau$, and by subscript 2 those normal to it. Then on dimensional considerations with allowance for the problem symmetry it is possible to show that for a perfect gas with specific heat ratio $\varkappa$:

$$[v_{sn}]=j\Phi_0/\rho_s^\circ,\quad [v_{s\tau 1}]=j\Phi_1/\rho_s^\circ,\quad [v_{s\tau 2}]=0 \tag{2.13}$$

$$\left(\begin{array}{c}\Phi_k=\Phi_k(\alpha_-, M_{n-}^r, v_{sn-}^r/v_{n-}^r, \rho_-^\circ/\rho_s^\circ, \varkappa),\ \alpha=|\mathbf{v}_{s\tau}-\mathbf{v}_\tau|/v_{sn}^r\\ M_n{}^r=v_n{}^r/a,\ v_n{}^r=v_n-D,\ v_{sn}^r=v_{sn}-D\end{array}\right)$$

where $\Phi_1(0,\ldots)=0$. The determination of functions Φ_k is a separate problem. Here we assume these functions and, also f or q in (1.3) to be known. In this case (2.13) closes (2.8). When $[v_{\tau 1}]\neq 0$, then in conformity with (2.13) and (2.8) $[v_{\tau 2}]=F_{\tau 2}=0$.

3. When considering discontinuities of the film type it is necessary to make additional assumptions about its structure. Their number is minimal when the flow parameters depend of time and only on a single space variable x, with the film represented by a plane normal to the x axis along which we direct the normal $\mathbf{n}$. Here, we consider flows of such type only, noting that although the assumptions made below are of a general nature, conditions at the film in the general and one-dimensional cases are not the same. This is associated with the transport of mass, momentum and energy along the film, which depend on its shape and the distribution of all surface properties.

The film is generated in consequence of intersections of particle trajectories, when "outdistancing" is inhibited. In the two-fluid approximation such inhibition is a property of the model (the medium of particles is considered to be a single-spped one), which on the face of it appears to be the consequence of its imperfection. Yet, even in the more complete three-fluid model which takes into account collisions of particles travelling at different speeds, narrow zones with particle density ρ_s of order ρ_s° appear at intersections of trajectories. If outside such zones $\rho_s \ll \rho_s^\circ$, they can be replaced by zero-thickness surfaces that carry surface mass, momentum, and energy. When $\rho_s^\circ \gg \rho^\circ$, we assume, as previously, that it is reasonable to associate the surface film properties only with particles, disregarding the gas that saturates the film. There is also no reason for taking into account gas accumulation in the film.

Let R_s be the surface density of particles in the film, $\mathbf{V}_s=\mathbf{n}D+\mathbf{V}_{s\tau}$ be the film velocity, and E_s the specific internal energy of its particles. On these assumptions we obtain from (1.2) that on a plane film

$$\frac{dR_s}{dt}=-\Delta_s^j,\quad \frac{d(R_sD)}{dt}=-\Delta^n,\quad \frac{d(R_s\mathbf{V}_{s\tau})}{dt}=-\boldsymbol{\Delta}^\tau \tag{3.1}$$

$$\frac{d\{R_s(2E_s+V_{s\tau}^2)\}}{dt}=-\Delta^i,\quad \Delta^j=0,\quad [(\rho/\rho^\circ)p+\rho(v_n-D)^2]=-F_n$$

$$[\rho(v_n-D)\mathbf{v}_\tau]=-\mathbf{F}_\tau,\quad j[2i+(v_n-D)^2+v_\tau{}^2]=$$
$$-2(R_sQ+\mathbf{F}_\tau\mathbf{V}_{s\tau})$$

where d/dt is the time derivative along the film trajectory, and $dx/dt=D$, $\Delta_s^j,\ldots$ are the same as in (2.1), although now $j_{s+}\neq j_{s-}$, R_sQ is the quantity of heat which the gas flowing through a film unit area transmits to it per unit of time. In the two-fluid model particles precipitate onto the film, and acquire its velocity and temperature. Such interaction, natural for fairly close packing of film particles, prevents the substitution of the law of conservation (1.4) of particle internal energy for the last of Eqs.(1.2).

Further assumptions about F_n, $\mathbf{F}_\tau$ and Q are necessary for closing (3.1). The film is assumed to be a layer of fairly tightly packed, hence strongly interacting particles that move as a single whole, having the same internal energy and to be homogeneous with respect to m_s. The quantity $m_s \equiv m_s^*$ or the film porosity $m^* = 1 - m_s^*$ are assume to be known functions of parameters (or be constants). In the considered here model any gas parameter φ, when passing through the film, undergoes a three-stage change: first, a sudden compression of the stream as m_s increases from m_{s-} to m_s^*, then interaction with the film of thickness $\lambda = R_s / m_s^* \rho_s^\circ$, followed by sudden expansion owing to the decrease of m_s from m_s^* to m_{s+}. The first and third stages are defined in terms of the contact discontinuity of the particles medium ($j \neq 0$, $j_s = 0$) considered in Sect.2. The definition of interaction with the homogeneous film in the approximation of, for instance, the simplest model of a porous medium also does not present any difficulties.

Let $[\varphi]^- = \varphi^- - \varphi_-$, $[\varphi]^\lambda = \varphi^+ - \varphi^-$, φ^- and φ^+ be the values of φ, when the left- and right-hand boundaries of the film are approached from its inside (the gas flows from left to right), and $[\varphi]^+ = \varphi_+ - \varphi^+$. Denoting similarly the inputs of F_n, $\mathbf{F}_\tau$ and Q, we obtain

$$[\varphi] = [\varphi]^- + [\varphi]^\lambda + [\varphi]^+, \quad F_n = F_n^- + F_n^\lambda + F_n^+, \quad \mathbf{F}_\tau = \mathbf{F}_\tau^\lambda, \quad Q = Q^\lambda \tag{3.2}$$

where the definition of quantities with indices plus and minus is the same as given in Sect.2. By virtue of the above the quantities with the superscript λ are known functions of film and gas parameters with the minus superscript. Omitting details, we shall show that these functions are obtained by integrating one-dimensional steady state equation of gas filtration in a homogeneous porous medium, which vanish when $R_s = 0$. Exact integration can be replaced the approximate one using the mean value theorem. As the result, $F^{u\lambda}$, $\mathbf{F}_\tau^\lambda$ and Q^λ are determined by the finite formulas $(v_n^- + v_n^+) / 2, \ldots$ instead of by integrals of local gas parameters.

4. Consider the one-dimensional problem of initial disintegration of an arbitrary discontinuity. We shall measure time from the instant of disintegration and x from the point of parameter discontinuity at $t = 0$. If τ is the minimal relaxation time determined by the terms $\mathbf{f}$, q, F_n, F_τ and Q in (1.5) and (3.2), the problem comprises the initial stage $0 < t \ll \tau$ in which in conformity with the definition of τ, these terms can be disregarded. Because of this, the determining parameters of time and length dimensions vanish, which makes the solution dependent on the self-similar variable $\xi = x / t$. The rays $\xi = \xi_k$ or the straight lines $x = \xi_k t$ with constants ξ_k, determined in the course of solution of the problem, divide the xt plane in zones of different structure. Besides zones of constant parameters (proper for each zone), the solution generally contains centered waves with continuous variation of parameters from one ray to another. Boundaries of each zone are defined either by discontinuity trajectories propagating at constant velocities $D = \xi_k$, or by the characteristics of system (1.5). On the film, as in the simpler self-similar problem /3/, $R_s = \beta t$, where β is a constant.

Centered waves are defined by conventional differential equations obtained from (1.5) with $\mathbf{f} = 0$ and $q = 0$ on the assumption that all parameters are functions of ξ only. In centered waves admitted by (1.5) $\mathbf{v}_\tau$, $\mathbf{v}_{s\tau}$, s and e_s are constant ($\mathbf{v}_\tau$ is the component of $\mathbf{v}$ normal to the x axis), and the remaining quantities satisfy the equations

$$\begin{gathered}(u - \xi)\{p' - \delta a^2 (\rho^\circ / \rho)\rho_s'\} + \rho^\circ a^2 u' = 0, \quad (u - \xi)\rho^\circ u' + p' = 0 \\ (u_s - \xi)\rho_s' + \rho_s u_s' = 0, \quad (u_s - \xi) u_s' + \varepsilon p' = 0\end{gathered} \tag{4.1}$$

where primes denote derivatives with respect to ξ, and u and u_s are projections of $\mathbf{v}$ and $\mathbf{v}_s$ on the x axis. Owing to the constancy of entropy, ρ° and a are known functions of p.

System (4.1) is linear and homogeneous with respect to u', p', u_s' and ρ_s', and has nontrivial solutions when the determinant of its coefficients is zero. Let $X = (u - \xi) / a$ and $\Delta = (u - u_s) / a$. Then this condition assumes the form (2.9). As already established, (2.9) also defines the velocities of low-intensity discontinuities, and in the case of one-dimensional unsteady flows it defines characteristics (1.5) different from the gas and particle trajectories. This and the results of Sect.2 show that centered waves can only consist of $c^\pm$ characteristics when $|\Delta| < \Delta_* \equiv (1 + \chi^{1/3})^{3/2}$. Here, in conformity with (2.10) we have $u - \xi = aX_{1,2} \approx \mp a$ when $\chi \ll 1$ hence, as implied by the first or second of Eqs.(4.1), $u' = \pm p' / (\rho^\circ a) + O(\chi)$, i.e. the formulas for centered waves in gas are accurate to $O(\chi)$. We call such waves $c^\pm$ rarefaction waves. If ρ_0 is a characteristic, for instance the initial density of gas, the increments of u_s and ρ_s are $\rho_0 \varepsilon = \rho_0 / \rho_s^\circ$ times smaller than the increments of u and p.

When $\Delta \geqslant \Delta_*$, (4.1) admits besides $c^\pm$-waves for which the above reasoning is valid, centered $c_s\pm$ -waves, for which in conformity with (2.11)

$$u_s - \xi = (X - \Delta)_{3,1} a \approx \mp a\Delta \sqrt{\chi/(\Delta^2 - 1)}$$

This and (4.1) imply that in centered $c_s\pm$ -waves

$$\xi - \xi_0 \approx \pm\left(\frac{3a\Delta\sqrt{\varepsilon\delta}}{\sqrt{\Delta^2-1}}\right)_0 (\sqrt{\rho_s} - \sqrt{\rho_{s0}}), \quad u_s - u_{s0} \approx \frac{2}{3}(\xi - \xi_0) \tag{4.2}$$

$$u - u_0 \approx \left\{\frac{a\Delta\delta}{\rho^\circ(\Delta^2-1)}\right\}_0 (\rho_{s0} - \rho_s), \quad p - p_0 \approx \left(\frac{a^2\Delta^2\delta}{\Delta^2-1}\right)_0 (\rho_s - \rho_{s0})$$

where the subscript zero denotes quantities at the wave origin, i.e. the variation of ρ_s is considerably greater than that of remaining parameters and ξ. The latter is important in investigations of $c_s\pm$ -waves in which particle density drops to zero, while the remaining quantities remain almost constant. By virtue of (2.11) the $c_s\pm$ -characteristic that corresponds to $\rho_s = 0$ and $\chi \sim \rho_s\varepsilon\delta = 0$ coincides with the trajectory of the particles.

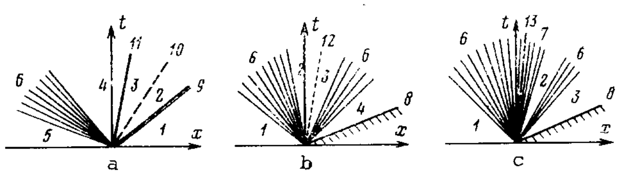

Fig.2

Theoretically discontinuities and centered waves enable us to construct solutions not only for the problem of disintegration of an arbitrary discontinuity but, also, for other self-similar problems, more exactly, for their initial stages $0 < t \ll \tau$. Three possible patterns of such flows are shown in Fig.2, where a corresponds to the problem of disintegration, and b and c to that of piston advance. The numerals 1 – 5 in Fig.2 indicate regions of uniform flow or quiescence, 6 of centered c^- -waves, 7 of c_s^- -waves, 8 piston trajectories, 9 those of the c^+ -discontinuity, 10 those of contact discontinuity in gas, 11 that of the film, 12 of the contact discontinuity in the particle medium, and 13 those of c^- and simultaneously of the c_s^- characteristic which separates the c^- and c^-_s waves. In Fig.2,b ρ_s vanishes abruptly in the particles contact discontinuity and in Fig.2,c it continuously approaches zero in the c_s^- -wave. In each specific case the solution scheme is determined conditions of the problem and its evolution requirements. The analysis of evolutionarity of system (1.5) and of the derived above discontinuities is the subject of a separate investigation.

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